A SIMPLE METHOD OF CONSTRUCTION OF SYMMETRICAL CONFOUNDED FACTORIAL DESIGNS*

By V. D. Thawani

Guahati University, Assam

1. Introduction

The concept of confounding in factorial experiments has been familiar for over a decade and the simpler designs, specially the symmetrical confounded designs involving a number of factors, all at the same level, have been worked out by Yates,[1] Nair,[2,3] Bose and Kishen[4] and others. The methods employed for the construction of these designs were, however, different in different cases. It was R. A. Fisher who drew attention first to the theory of Abelian groups and the relations recognisable in the choice of interactions for confounding. In his first paper,[5] he developed the theory to cover only cases involving factors at two levels each. This was extended by him later[6] to (i) factors involving any prime number of alternatives and (ii) to the case in which the number of levels is a power of a prime. The same method was also employed by Finney[7] in developing the theory of fractional replications of factorial arrangements. It has been the experience of the author, that among the several methods of approaching this problem, the application of the properties of Abelian groups, offers a particularly convenient and simple way of obtaining the various alternative layouts of a symmetrical confounded system, specially when neither the number of factors nor the number of alternatives is large. The essence of the material contained in Sections 2–5 of this paper is to be found in Finney's paper[7] and Sections 6 and 7 are extensions. The method is further developed in Sections 8–10 to cover the case of factors at four levels. The object of this paper is to bring together in one place for the benefit of students and others a few designs with factors at three and four levels, by making use of this unified procedure based upon the theory of Abelian groups. The method is general and could be extended to other symmetrical cases as well. It would indeed be interesting if the method of groups could be employed for the construction of non-symmetrical factorial designs, but there are obvious difficulties in the way.

* Received March 1951—*Ed.*

in which $b = mn$ and $a = m + n$, m and n being the number of points on the sides of the lattice.

Similarly by substituting the values of a, A, B and C given in Table II the 4th cumulant for the distribution of B-w and w-B joins and also for the total number of joins between points of different colours and also for the total number of joins between points of different colours can be written.

5. Summary

A rigorous proof for the author's result for calculating the factorial moments of distributions arising in Markoff chains has been given. A new simple method for obtaining the cumulants of such distributions has also been developed.

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(Vol. IV, No. 1, p. 61)

Eighth row of the table—*Read* " 18 " *in place of* " 17 ".

3rd and 4th lines from the bottom—*Read* " 18 " *in place of* " 17 ".

2. The 'Effect' and the 'Treatment' Groups

Let the n factors $A, B, C, \ldots$ be taken at p levels each, where p is a prime. Then the p^n elements

$$I, A, B, A^2B, C, \qquad (2.1)$$

form an Abelian group of order p^n, called the 'Effect' group, with I as the identity. The rule of combination of the elements of (2.1) is multiplication and the elements satisfy the usual group conditions

$$\left.\begin{array}{l} A^p = B^p = C^p = \ldots = I, \\ IA = A = AI, \text{ etc.} \end{array}\right\} \qquad (2.2)$$

It follows that the product of any number of elements of (2.1) is an element of (2.1).

Again, if (1) denotes a treatment combination with all the factors at zero level and (a) differs from it in that it is at level one of A, (a^2b) differs from it in that it is at level two of A and one of B and so on, then the p^n elements (dropping the brackets)

$$1, a, b, a^2b, c, \qquad (2.3)$$

form an Abelian group of order p^n, called the 'Treatment' group, with 1 as the identity. Here too the rule of combination is multiplication and the elements satisfy conditions similar to (2.2), *viz.*,

$$\left.\begin{array}{l} a^p = b^p = c^p = \ldots = 1, \\ 1.a = a = a.1, \text{ etc.} \end{array}\right\} \qquad (2.4)$$

It follows again that the product of any number of elements of (2.3) is an element of (2.3).

Elements $A^\alpha.B^\beta.C^\gamma\ldots$ and $a^{\alpha'}.b^{\beta'}.c^{\gamma'}\ldots$ will be called *orthogonal* if

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' + \ldots = 0 \pmod{p} \qquad (2.5)$$

A *sub-group* of (2.1) or (2.3) consists of elements of it, satisfying the multiplicative property of the parent group. Every sub-group must necessarily contain the identity. As an example, for a 3^2 design, a sub-group of the effect group is

$$(I, AB^2, A^2B) \qquad (2.6)$$

which is of order 3. In general, a sub-group of a group of order p^n, where p is prime, consists of p^α elements, α being a positive integer less than n.

Two sub-groups are said to be orthogonal to each other if every element of one sub-group is orthogonal to every element of the other sub-group. For example, the sub-group orthogonal to (2.6) is

$$(1, ab, a^2b^2) \tag{2.7}$$

In the theory of confounded designs, a sub-group like (2.6) will be called the confounding sub-group, specifying the interactions confounded, whereas (2.7) will be called the intra-block sub-group orthogonal to (2.6), consisting of all those elements of the treatment group which are orthogonal to the elements of the confounding sub-group. If the group is of order p^n (p prime) and the confounding sub-group is of order p^α, the intra-block sub-group will be of order $p^{(n-\alpha)}$.

3. 3^2 Design

If A, B are two factors, at three levels each, there are 8 d.f. for treatment effects, out of which 4 d.f. belong to the main effects and 4 d.f. to the interaction of A and B. The sub-group

$$(I, A, A^2) \tag{3.1}$$

is orthogonal to a set of treatment combinations, not containing a, *viz.*,

$$(1, b, b^2) \tag{3.2}$$

If, therefore, the three treatment combinations (3.2) be allocated to one sub-block and those obtained by multiplying the elements of (3.2) by a, a^2, viz.,

$$(a, ab, ab^2) \tag{3.3}$$

and

$$(a^2, a^2b, a^2b^2) \tag{3.4}$$

to two other sub-blocks, then the contrasts between the three sub-blocks (3.2), (3.3), (3.4) are equivalent to the contrasts between the three levels of the factor A. Thus the sub-group (3.1) may be said to represent the main effect of A.

The interaction elements are

$$AB, A^2B, AB^2, A^2B^2 \tag{3.5}$$

which in conjunction with I, automatically form themselves into two sub-groups, *viz.*,

$$(I, AB^2, A^2B) \tag{3.6}$$

and

$$(I, AB, A^2B^2) \tag{3.7}$$

which in the notation of Yates may be called the I and J components of the interaction $A \times B$. We have

$$I(A \times B) \equiv (I, AB^2, A^2B) \tag{3.8}$$

$$J(A \times B) \equiv (I, AB, A^2B^2) \tag{3.9}$$

and 2 d.f. correspond to each of I and J.

If confounding is required, which is hardly necessary for a 3^2 design, main effects must be left free and one of the sub-groups (3.8) or (3.9) may be chosen as the confounding sub-group to give a design with three sub-blocks of three plots each. Thus if (3.8) be selected as the confounding sub-group, the intra-block sub-group of treatment combinations orthogonal to it is

$$(1, ab, a^2b^2) \tag{3.10}$$

If the treatment combinations (3.10) be allocated to one sub-block and those obtained by multiplying (3.10) by a and a^2, *viz.*,

$$(a, a^2b, b^2) \text{ and } (a^2, b, ab^2) \tag{3.11}$$

to two other sub-blocks, then the 2 d.f. corresponding to $I(A \times B)$ are confounded with the sub-blocks.

This method of division of 4 d.f. for interaction into two orthogonal sets of 2 d.f. each, is fruitful as it is capable of an extension to the case of several factors at three levels each.

4. 3^3 Design

In this case, it is desirable to leave the 6 d.f. for the main effects and 12 d.f. for two-factor interactions free, and confound some 2 d.f. for the three-factor interaction to obtain three sub-blocks of nine plots each. The elements of the effect group corresponding to the three-factor interaction are

$$ABC,\ A^2BC,\ AB^2C,\ A^2B^2C,\ ABC^2,\ A^2BC^2,\ AB^2C^2,\ A^2B^2C^2 \tag{4.1}$$

Along with the identity I, they automatically form themselves into four sub-groups; *viz.*, (I, AB^2C^2, A^2BC), (I, AB^2C, A^2BC^2), (I, ABC^2, A^2B^2C) and $(I, ABC, A^2B^2C^2)$, which in the notation of Yates may be identified with W, X, Y, Z respectively. Moreover from the rules (3.8) and (3.9) we may write

$$W \equiv (I, AB^2C^2, A^2BC) \equiv I\{I(A \times B) \times C\} \tag{4.2}$$

$$X \equiv (I, AB^2C, A^2BC^2) \equiv J\{I(A \times B) \times C\} \tag{4.3}$$

$$Y \equiv (I, ABC^2, A^2B^2C) \equiv I\{J(A \times B) \times C\} \tag{4.4}$$

$$Z \equiv (I, ABC, A^2B^2C^2) \equiv J\{J(A \times B) \times C\} \tag{4.5}$$

Any 2 d.f. corresponding to one of the four orthogonal sets (4.2) to (4.5) may be confounded. Thus if Y be confounded, the intra-block sub-group of treatment combinations consists of nine elements orthogonal to (4.4). The required sub-group is

$$(1, abc^2, a^2b, a^2b^2c, ab^2, b^2c^2, a^2c^2, ac, bc) \qquad (4.6)$$

Actually if any two independent elements of (4.6) orthogonal to the elements of (4.4) are found, the whole sub-group (4.6) can be generated. Moreover the orthogonality of these two elements may be tested only for one element of (4.4) since it will then automatically hold for the other element. In general, for a 3^n design, if the confounding sub-group consists of 3^k elements, generated by any k independent elements, the intra-block sub-group consists of 3^{n-k} elements, generated by any $(n - k)$ independent elements, each of which is orthogonal to the k elements generating the confounding sub-group. If the treatment combinations of (4.6) be allotted to one sub-block, the two sub-blocks obtained by multiplying (4.6) by a, a^2 are

$$\left.\begin{array}{l}(a, a^2bc^2, b, b^2c, a^2b^2, ab^2c^2, c^2, a^2c, abc) \\ (a^2, bc^2, ab, ab^2c, b^2, a^2b^2c^2, ac^2, c, a^2bc)\end{array}\right\} \qquad (4.7)$$

and

If (4.6) and (4.7) form the three sub-blocks, then 2 d.f. for Y are confounded.

In the same way, if 2 d.f. or W are confounded, the three sub-blocks are

$$\left.\begin{array}{l}(1, ab, a^2b^2, a^2bc, b^2c, ac, ab^2c^2, bc^2, a^2c^2) \\ (a, a^2b, b^2, bc, ab^2c, a^2c, a^2b^2c^2, abc^2, c^2) \\ (a^2, b, ab^2, abc, a^2b^2c, c, b^2c^2, a^2bc^2, ac^2)\end{array}\right\} \qquad (4.8)$$

If 2 d.f. for X are confounded, the three sub-blocks are

$$\left.\begin{array}{l}(1, ab, a^2b^2, a^2c, bc, ab^2c, ac^2, b^2c^2, a^2bc^2) \\ (a, a^2b, b^2, c, abc, a^2b^2c, a^2c^2, ab^2c^2, bc^2) \\ (a^2, b, ab^2, ac, a^2bc, b^2c, c^2, a^2b^2c^2, abc^2)\end{array}\right\} \qquad (4.9)$$

Finally if two d.f. for Z are confounded, the three sub-blocks are

$$\left.\begin{array}{l}(1, ab^2, a^2b, ac^2, a^2b^2c^2, bc^2, a^2c, abc, b^2c) \\ (a, a^2b^2, b, a^2c^2, b^2c^2, abc^2, c, a^2bc, ab^2c) \\ (a^2, b^2, ab, c^2, ab^2c^2, a^2bc^2, ac, bc, a^2b^2c)\end{array}\right\} \qquad (4.10)$$

A balanced design can be obtained with four replications, confounding W, X, Y, Z, one in each replication and thereby preserving three-fourths the information even on the three-factor interaction, the main effects and two-factor interactions being unconfounded.

3^3 design in nine blocks of three plots each is unsuitable but its chief features may be mentioned. Here 8 d.f. are confounded and it is desirable to leave the main effects free and confound as many degrees of freedom due to the three-factor interaction as possible. However if W and X be both confounded, the main effect of C is also confounded. Therefore, consistent with the condition that the main effects are to be kept free, only one set of 2 d.f. for the three-factor interaction corresponding to W or X or Y or Z can be confounded. The remaining 6 d.f. to be confounded necessarily belong to two-factor interactions. With W, $I(A\times B)$ cannot be confounded as that would mean confounding the main effect of C also. Following degrees of freedom can be simultaneously confounded:

W or $I\{I(A\times B)\times C\}$, together with $J(A\times B)$, $J(A\times C)$, $I(B\times C)$ (4.11)

X or $J\{I(A\times B)\times C\}$, together with $J(A\times B)$, $I(A\times C)$, $J(B\times C)$ (4.12)

Y or $I\{J(A\times B)\times C\}$, together with $I(A\times B)$, $J(A\times C)$, $J(B\times C)$ (4.13)

Z or $J\{J(A\times B)\times C\}$, together with $I(A\times B)$, $I(A\times C)$, $I(B\times C)$ (4.14)

If the 8 d.f. due to (4.11) be confounded, the confounding sub-group is

$$(I, AB, A^2B^2, AC, A^2C^2, B^2C, BC^2, A^2BC, AB^2C^2) \qquad (4.15)$$

The intra-block sub-group of treatment combinations orthogonal to (4.15) is

$$(1, a^2bc, ab^2c^2) \qquad (4.16)$$

The elements of (4.16) form one sub-block and the remaining eight sub-blocks, obtained in the usual manner by multiplication of its elements by suitable elements of the treatment group, are

$$(a, bc, a^2b^2c^2) \qquad (4.17)$$
$$(a^2, abc, b^2c^2) \qquad (4.18)$$
$$(b, a^2b^2c, ac^2) \qquad (4.19)$$
$$(b^2, a^2c, abc^2) \qquad (4.20)$$
$$(c, a^2bc^2, ab^2) \qquad (4.21)$$
$$(c^2, a^2b, ab^2c) \qquad (4.22)$$
$$(ab, b^2c, a^2c^2) \qquad (4.23)$$
$$(a^2b^2, ac, bc^2) \qquad (4.24)$$

(4.16) to (4.24) give the nine sub-blocks of three plots each, confounding the eight degrees of freedom given by (4.11) or (4.15).

With four replications, a balanced design can be obtained in which each of I and J components of the two-factor interactions is confounded in two replications only, while each of the components W, X, Y, Z of the three-factor interaction is confounded in one replication only. Thus half the information on the two-factor interactions and a quarter on the three-factor interaction is lost, while the main effects are preserved.

5. 3^4 DESIGN

The design may usefully be arranged in nine sub-blocks of nine plots each, per replication, confounding the 8 d.f. due to three-factor interactions only. The 8 d.f. for $A\times B\times C$ form themselves into the four sub-groups (4.2) to (4.5). Similar sub-groups exist for the 8 d.f. corresponding to the interaction $A\times B\times D$. With $Z(A\times B\times C)$, can be associated only $W(A\times B\times D)$ or $X(A\times B\times D)$ in the confounding sub-group as $Y(A\times B\times D)$ or $Z(A\times B\times D)$ leads to the confounding of two-factor interactions also. Thus for each of W, X, Y, Z components of $A\times B\times C$, there are two possible ways of combining the 2 d.f. out of $A\times B\times D$, giving eight possible combinations in all. It is seen that the degrees of freedom confounded for $B\times C\times D$ and $A\times C\times D$ are then automatically fixed up. The eight possible confounding sub-groups may be given below:

$$\left.\begin{array}{l}(I,\ ABC,\ A^2B^2C^2,\ A^2BD,\ AB^2D^2,\ AC^2D,\ A^2CD^2,\ B^2CD,\ BC^2D^2)\\ \text{confounding}\\ Z(A\times B\times C),\ W(A\times B\times D),\ X(A\times C\times D),\ W(B\times C\times D)\end{array}\right\}\quad(5.1)$$

$$\left.\begin{array}{l}(I,\ ABC,\ A^2B^2C^2,\ AB^2D,\ A^2BD^2,\ AC^2D^2,\ A^2CD,\ B^2CD^2,\ BC^2D)\\ \text{confounding}\\ Z(A\times B\times C),\ X(A\times B\times D),\ W(A\times C\times D),\ X(B\times C\times D)\end{array}\right\}\quad(5.2)$$

$$\left.\begin{array}{l}(I,\ A^2BC,\ AB^2C^2,\ ABD,\ A^2B^2D^2,\ ACD^2,\ A^2C^2D,\ B^2CD,\ BC^2D^2)\\ \text{confounding}\\ W(A\times B\times C),\ Z(A\times B\times D),\ Y(A\times C\times D),\ W(B\times C\times D)\end{array}\right\}\quad(5.3)$$

$$\left.\begin{array}{l}(I,\ A^2BC,\ AB^2C^2,\ A^2B^2D,\ ABD^2,\ ACD,\ A^2C^2D^2,\ BC^2D,\ B^2CD^2)\\ \text{confounding}\\ W(A\times B\times C),\ Y(A\times B\times D),\ Z(A\times C\times D),\ X(B\times C\times D)\end{array}\right\}\quad(5.4)$$

$$\left.\begin{array}{l}(I,\ AB^2C,\ A^2BC^2,\ ABD,\ A^2B^2D^2,\ AC^2D^2,\ A^2CD,\ BCD^2,\ B^2C^2D)\\ \text{confounding}\\ X(A\times B\times C),\ Z(A\times B\times D),\ W(A\times C\times D),\ Y(B\times C\times D)\end{array}\right\}\quad(5.5)$$

$$\left.\begin{array}{l}(I,\ AB^2C,\ A^2BC^2,\ A^2B^2D,\ ABD^2,\ AC^2D,\ A^2CD^2,\ BCD,\ B^2C^2D^2)\\ \text{confounding}\\ X(A\times B\times C),\ Y(A\times B\times D),\ X(A\times C\times D),\ Z(B\times C\times D)\end{array}\right\}\quad(5.6)$$

$$\left.\begin{array}{l}(I, ABC^2, A^2B^2C, A^2BD, AB^2D^2, ACD, A^2C^2D^2, BCD^2, B^2C^2D) \\ \text{confounding} \\ \quad Y(A\times B\times C),\ W(A\times B\times D),\ Z(A\times C\times D),\ Y(B\times C\times D)\end{array}\right\}(5.7)$$

$$\left.\begin{array}{l}(I, ABC^2, A^2B^2C, AB^2D, A^2BD^2, ACD^2, A^2C^2D, BCD, B^2C^2D^2) \\ \text{confounding} \\ \quad Y(A\times B\times C),\ X(A\times B\times D),\ Y(A\times C\times D),\ Z(B\times C\times D)\end{array}\right\}(5.8)$$

Any one of the sub-groups (5.1) to (5.8) may be chosen as the confounding sub-groups. The intra-block sub-group of nine treatment combinations orthogonal to it can be easily written down in the usual manner. It is generated by just two independent elements. For example, if (5.1) be selected as the confounding sub-group, the elements abc, a^2bd are orthogonal to every element of it. The control sub-block being generated by these elements is

$$(1, abc, a^2bd, a^2b^2c^2, b^2cd, ac^2d, ab^2d^2, a^2cd^2, bc^2d^2) \qquad (5.9)$$

The remaining eight sub-blocks are then written out with the help of (5.9) in the usual manner by multiplication of its elements by any elements of the treatment group. For ready reference they are given below:

$$\left.\begin{array}{l}(a, a^2bc, bd, b^2c^2, ab^2cd, a^2c^2d, a^2b^2d^2, cd^2, abc^2d^2) \\ (a^2, bc, abd, ab^2c^2, a^2b^2cd, c^2d, b^2d^2, acd^2, a^2bc^2d^2) \\ (b, ab^2c, a^2b^2d, a^2c^2, cd, abc^2d, ad^2, a^2bcd^2, b^2c^2d^2) \\ (b^2, ac, a^2d, a^2bc^2, bcd, ab^2c^2d, abd^2, a^2b^2cd^2, c^2d^2) \\ (c, abc^2, a^2bcd, a^2b^2, b^2c^2d, ad, ab^2cd^2, a^2c^2d^2, bd^2) \\ (c^2, ab, a^2bc^2d, a^2b^2c, b^2d, acd, ab^2c^2d^2, a^2d^2, bcd^2) \\ (d, abcd, a^2bd^2, a^2b^2c^2d, b^2cd^2, ac^2d^2, ab^2, a^2c, bc^2) \\ (d^2, abcd^2, a^2b, a^2b^2c^2d^2, b^2c, ac^2, ab^2d, a^2cd, bc^2d)\end{array}\right\}(5.10)$$

If the design be arranged in nine sub-blocks given by (5.9) and (5.10), then the 8 d.f. represented by (5.1) are confounded. In the same way, we can write down the nine sub-blocks, when any one of (5.2) to (5.8) is chosen as the confounding sub-group. A balanced design can evidently be obtained with four replications if (5.1), (5.4), (5.5) and (5.8) are confounded, one in each replication. In such a design, each of W, X, Y, Z for each of the four three-factor interactions is confounded once only, thus yielding three-fourths the information on each of the three-factor interactions, while preserving all other effects. In the same way, another balanced design with four replications, confounding (5.2), (5.3), (5.6) and (5.7), one in each replication, can be obtained.

6. 3^n Design in 3 Sub-Blocks of 3^{n-1} Plots Each

There are 2^n d.f. for the n-factor interaction, which by generalisation of the results of the preceding sections, can be broken up into 2^{n-1} orthogonal sets of 2 d.f. each. For example, when $n=3$, the 2^3 d.f. for the three-factor interaction $A \times B \times C$ can be broken up into 2^2 sets, viz., W, X, Y, Z of 2 d.f. each. If a fourth factor D be introduced, then with $W(A \times B \times C)$ can be associated two sub-groups

$$\left. \begin{aligned} W_1 &\equiv (I,\ A^2BCD,\ AB^2C^2D^2) \\ W_2 &\equiv (I,\ A^2BCD^2,\ AB^2C^2D) \end{aligned} \right\} \qquad (6.1)$$

Similarly, two sub-groups X_1, X_2 correspond to X; Y_1, Y_2 to Y, and Z_1, Z_2 to Z, giving 2^3 sub-groups in all of 2 d.f. each. If a fifth factor E be introduced, two sub-groups correspond to each of W_1, W_2, X_1, X_2, Y_1, Y_2, Z_1, Z_2, giving 2^4 sub-groups of 2 d.f. each. Proceeding in this way, 2^{n-1} orthogonal sets of 2 d.f. each, corresponding to the highest order interaction can be obtained. Any one of these 2^{n-1} sets may be confounded and the sub-block with control treatment, containing 3^{n-1} elements of the treatment group orthogonal to it can be written out, the remaining two sub-blocks being then written out in the usual manner.

7. 3^5 Design in 3^2 Blocks of 3^3 Plots Each

The confounding sub-groups of Section 5 could be modified by the introduction of a fifth factor E. Thus out of (5.1) could be obtained two confounding sub-groups

$$(I,\ ABC,\ A^2B^2C^2,\ A^2BDE,\ AB^2D^2E^2,\ AC^2DE,\ A^2CD^2E^2,\ B^2CDE,\ BC^2D^2E^2) \qquad (7.1)$$

$$(I,\ ABC,\ A^2B^2C^2,\ A^2BDE^2,\ AB^2D^2E,\ AC^2DE^2,\ A^2CD^2E,\ B^2CDE^2,\ BC^2D^2E) \qquad (7.2)$$

In (7.1) or (7.2), 2 d.f. confounded belong to three-factor interaction, while the remaining 6 d.f. confounded belong to four-factor interactions. Thus, if 2 d.f. belonging to $A \times B \times C$ are confounded, sixteen possible confounding sub-groups are thereby obtained. But as there are 10 ways of selecting three factors out of five, there are in all 160 possible ways of arranging a 3^5 design in 3^2 blocks of 3^3 plots each, confounding in each arrangement, 2 d.f. due to a three-factor interaction and 6 d.f. due to a four-factor interaction, all other effects being unconfounded.

8. Factors at Four Levels

The methods of preceding sections could be applied to any case in which the number of levels is a prime. The procedure has to be slightly modified if the number of levels is a power of a prime and may be illustrated by considering factors at 2^2 levels.

The 3 d.f. corresponding to the main effect of A may be represented by the group

$$(I, A_1, A_2, A_3) \tag{8.1}$$

with the conditions

$$A_1^2 = A_2^2 = A_3^2 = I, A_1A_2 = A_3, A_1A_3 = A_2, A_2A_3 = A_1 \tag{8.2}$$

which are obviously consistent.

If there are n factors at four levels each, the effect group and the treatment group consist of 4^n elements each. In the treatment group, an element such as $a_rb_sc_t\ldots$ will denote a treatment combination with the factor A at level r, B at level s, C at level t, etc. ($r, s, t\ldots = 0, 1, 2, 3$). Actually, a letter with suffix zero is dropped altogether. The orthogonality (mod. 2) of an element of the effect group with an element of the treatment group follows at once if the two elements have an even number of letters in common, after they are expressed with suffixes 1 and 2 by using (8.2) and similar relations. Thus the elements A_1B_3 and a_2b_3 expressed as $A_1B_1B_2$ and $a_2b_1b_2$ are clearly orthogonal, as they have two elements in common.

9. 4^2 Design

Let A, B be two factors at four levels each. The effect group consists of 16 elements obtained by multiplying out the elements of (I, A_1, A_2, A_3) and (I, B_1, B_2, B_3). If confounding is required at all, it is desirable to leave the main effects free and confound only some of 9 d.f. for the interaction, which are represented by the elements

$$(I, A_1B_1, A_1B_2, A_1B_3, A_2B_1, A_2B_2, A_2B_3, A_3B_1, A_3B_2, A_3B_3) \tag{9.1}$$

The elements of (9.1) automatically form themselves into three orthogonal components P_1, P_2, P_3 (say) of 3 d.f. each, represented by the sub-groups

$$P_1(A\times B) \equiv (I, A_1B_1, A_2B_2, A_3B_3) \tag{9.2}$$

$$P_2(A\times B) \equiv (I, A_1B_2, A_2B_3, A_3B_1) \tag{9.3}$$

$$P_3(A\times B) \equiv (I, A_1B_3, A_2B_1, A_3B_2) \tag{9.4}$$

(9.1) can alternatively be broken up into another set of three orthogonal components Q_1, Q_2, Q_3 (say) of 3 d.f. each, represented by the sub-groups

$$Q_1(A\times B) \equiv (I, A_1B_1, A_2B_3, A_3B_2) \tag{9.5}$$

$$Q_2(A\times B) \equiv (I, A_1B_2, A_2B_1, A_3B_3) \tag{9.6}$$

$$Q_3(A\times B) \equiv (I, A_1B_3, A_2B_2, A_3B_1) \tag{9.7}$$

The design could be arranged in four sub-blocks of four plots each, per replication, by choosing the confounding sub-group any one of (9.2) to (9.7). For example, if P_2 be confounded, to find the sub-block containing the control treatment, P_2 must first be expressed as

$$(I,\ A_1B_2,\ A_2B_1B_2,\ A_1A_2B_1) \tag{9.8}$$

in view of (8.2); and the sub-group of the treatment group orthogonal to (9.8) is

$$(1,\ a_2b_1,\ a_1b_1b_2,\ a_1a_2b_2) \tag{9.9}$$

which reduces to

$$(1,\ a_2b_1,\ a_1b_3,\ a_3b_2) \tag{9.10}$$

Each element of (9.9) has an even number of letters in common with each element of (9.8), showing that the two sub-groups are orthogonal. (9.10) gives the treatment combinations of the control sub-block, from which the remaining three sub-blocks can be written out in the usual manner. For ready reference they are given below:

$$\left.\begin{array}{l}(a_1,\ a_3b_1,\ b_3,\ a_2b_2)\\(a_2,\ b_1,\ a_3b_3,\ a_1b_2)\\(a_3,\ a_1b_1,\ a_2b_3,\ b_2)\end{array}\right\} \tag{9.11}$$

If (9.10) and (9.11) form the four sub-blocks, then 3 d.f. for $P_2\,(A\times B)$ will be confounded with sub-blocks. Similarly, we can get the design, when the 3 d.f. confounded belong to P_1, P_3, Q_1, Q_2, Q_3.

Three replications could give a balanced design, confounding the set P_1, P_2, P_3, one in each replication, thereby preserving two-thirds the information on each component. Another balanced design of three replications could be obtained by confounding the set Q_1, Q_2, Q_3.

10. 4^3 DESIGN

The design can usefully be arranged in four sub-blocks of sixteen plots each, per replication, confounding a set of 3 d.f. belonging solely to the three-factor interaction. If 3 d.f. due to $P_1\,(A\times B)$, given by (9.2) be taken, the third factor C could be introduced with it and the six possible confounding sub-groups are

$$\left.\begin{array}{l}P_1\,\{P_1\,(A\times B)\times C\},\ P_2\,\{P_1\,(A\times B)\times C\},\ P_3\,\{P_1\,(A\times B)\times C\}\\Q_1\,\{P_1\,(A\times B)\times C\},\ Q_2\,\{P_1\,(A\times B)\times C\},\ Q_3\,\{P_1\,(A\times B)\times C\}\end{array}\right] \tag{10.1}$$

Similar sets of confounding sub-groups could be taken with each of (9.3) to (9.7). It appears, therefore, that 36 confounding sub-groups could be chosen in all. The control sub-block corresponding to any given confounding sub-group, consists of 16 elements of the treatment group, orthogonal to it. Thus let the confounding sub-group be

$$P_2\{P_1(A\times B)\times C\}$$
$$\equiv (I,\ A_1B_1C_2,\ A_2B_2C_3,\ A_3B_3C_1)$$
$$\equiv (I,\ A_1B_1C_2,\ A_2B_2C_1C_2,\ A_1A_2B_1B_2C_1) \qquad (10.2)$$

To find the intra-block sub-group of 16 treatment combinations orthogonal to (10.2), only four independent elements, besides the identity, having an even number of letters in common with the elements of (10.2) are required. They may be taken as

$$a_1b_1,\ a_2b_2,\ a_1a_2c_2,\ b_1c_1c_2 \qquad (10.3)$$

which along with the identity generate the sub-group

$$\begin{pmatrix} 1,\ a_1b_1,\ a_2b_2,\ a_1a_2c_2,\ b_1c_1c_2,\ a_1a_2b_1b_2,\ a_2b_1c_2, \\ a_1c_1c_2,\ a_1b_2c_2,\ a_2b_1b_2c_1c_2,\ a_1a_2b_1c_1, \\ b_1b_2c_2,\ a_1a_2b_2c_1c_2,\ a_2c_1,\ a_1b_1b_2c_1,\ b_2c_1 \end{pmatrix} \qquad (10.4)$$

which becomes

$$\begin{pmatrix} 1,\ a_1b_1,\ a_2b_2,\ a_3c_2,\ b_1c_3,\ a_3b_3,\ a_2b_1c_2,\ a_1c_3, \\ a_1b_2c_2,\ a_2b_3c_3,\ a_3b_1c_1,\ b_3c_2,\ a_3b_2c_3,\ a_2c_1,\ a_1b_3c_1,\ b_2c_1 \end{pmatrix} \qquad (10.5)$$

The elements of (10.5) form the control sub-block and the other three sub-blocks could be written out from it in the usual manner. For ready reference they are given below:

$$\left.\begin{array}{l} \begin{pmatrix} a_1,\ b_1,\ a_3b_2,\ a_2c_2,\ a_1b_1c_3,\ a_2b_3,\ a_3b_1c_2,\ c_3, \\ b_2c_2,\ a_3b_3c_3,\ a_2b_1c_1,\ a_1b_3c_2,\ a_2b_2c_3,\ a_3c_1,\ b_3c_1,\ a_1b_2c_1 \end{pmatrix}, \\ \begin{pmatrix} a_2,\ a_3b_1,\ b_2,\ a_1c_2,\ a_2b_1c_3,\ a_1b_3,\ b_1c_2,\ a_3c_3, \\ a_3b_2c_2,\ b_3c_3,\ a_1b_1c_1,\ a_2b_3c_2,\ a_1b_2c_3,\ c_1,\ a_3b_3c_1,\ a_2b_2c_1 \end{pmatrix}, \\ \begin{pmatrix} a_3,\ a_2b_1,\ a_1b_2,\ c_2,\ a_3b_1c_3,\ b_3,\ a_1b_1c_2,\ a_2c_3, \\ a_2b_2c_2,\ a_1b_3c_3,\ b_1c_1,\ a_3b_3c_2,\ b_2c_3,\ a_1c_1,\ a_2b_3c_1,\ a_3b_2c_1 \end{pmatrix}. \end{array}\right\} \qquad (10.6)$$

Thus if (10.5) and (10.6) form the four sub-blocks of a design, the 3 d.f. for $P_2\{P_1(A\times B)\times C\}$ are confounded. Similarly, we can form the design when any one of certain 36 sub-groups is chosen as the confounding sub-group. Nine replications give a balanced design in which all the 27 d.f. due to the three-factor interaction are confounded and there are four such balanced designs.

SUMMARY

The theory of Abelian groups has been applied to the construction of confounded 3^n designs. The cases $n = 2, 3, 4$ have been

studied in some detail. It will be found that this method of construction is considerably simpler than the methods based upon the use of orthogonal Latin squares and involves no strain upon the memory. 2^n designs have been left out of discussion as they have been treated fully by this method by R. A. Fisher. The method is extended to 4^n designs and as an illustration, the cases $n = 2, 3$ have been discussed.

Acknowledgements

The material of this paper was ready as early as 1948 and was prepared by me while I was lecturing to the Diploma Classes in Agricultural Statistics at the Indian Council of Agricultural Research, New Delhi. I am deeply indebted to Dr. P. V. Sukhatme, Statistical Adviser, I.C.A.R., for his general guidance and encouragement. I am also grateful to Shri V. N. Amble, Statistician, I.C.A.R., with whom I discussed the results from time to time and who has given me valuable suggestions in the writing out of this paper. The simplified notation given in Section 8 is due to him.

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